Mech 463 Mechanical Vibrations

Lab Experiment

Natural Frequency and Radius of Gyration Measurement using a Bifilar Pendulum

G.S. Schajer
University of British Columbia
Vancouver, Canada

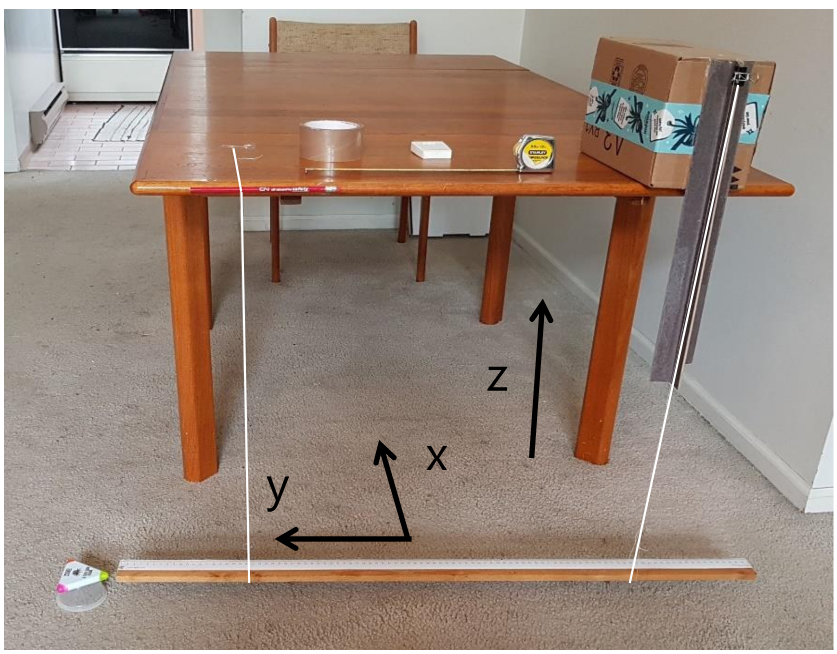

Figure 1. Bifilar Pendulum.

1. Objectives

1. To measure the natural frequencies and mode shapes of a bifilar pendulum.
2. To gain insight into the vibration of a multiple degree-of-freedom system.
3. To observe how the string separation and pendulum offset influence the natural frequencies and mode shapes of a bifilar pendulum.
4. To observe and learn about the Center of Percussion of an object.
5. To observe how unequal string lengths disrupt the pendulum symmetry and cause modal coupling and avoided crossing of natural frequencies.
6. To measure the radius of gyration of regular and irregular shaped objects.

2. Experiment Background

This is a fully "hands-on" experiment; you are asked not only to do the experiment but also to make the apparatus. The lab is designed as a "capstone" experiment and is more extended and challenging compared with a conventional lab. You can work at home and you are encouraged to use your imagination and to have some fun making/improvising the needed items. This experience of designing and building your own measurement equipment will be a good preparation for your future professional work. Then you will be doing things that have never been done before and you will need to be skillful to design and build your own specialized lab equipment.

3. Prelab Preparation

This will be a challenging lab, so it is important to do some serious preparation before starting. Two types of preparation are required, theoretical and practical.

3.1. Theoretical Preparation

Carefully read the Introduction and Theory sections of this instruction manual **before** starting anything else. Some ideas presented there will be familiar and many will be new. It is important that you clearly understand all the theoretical issues so that you can effectively do the next step, which is to design and build your own apparatus to explore them.

3.2. Practical Preparation

After reading and understanding the Theory, your next task is to gather together materials to build your own bifilar pendulum at home. You will need the following items:

1. A tape measure, either the steel type used by carpenters, or the fabric type used by tailors. Both items are readily available at dollar stores.
2. A metre ruler is ideal if available. Alternatively, a uniform stick of wood will work, preferably but not essentially cut to one metre long. You can seek materials at a hardware store or dollar store. If not using a ruler, attach an extra fabric tape measure along the length of your stick. Align your markings with the center of mass (balance point) of your stick, rather than the ends.
3. String to support your pendulum. Dental floss works well for this purpose.
4. A sturdy table with free space between the legs within which you can set up your bifilar pendulum.
5. An empty cardboard box that is rigid and in good condition. A convenient size is about 25 x 20 x 15cm^3, but any approximately similar size will work well.
6. Two pieces of stiff cardboard, e.g., from a strong packing box, each ~50x6cm^2.
7. A medium size binder clip. (These are usually black and have attached arms.)
8. Packing tape to secure strings and cardboard pieces.
9. A pencil, preferably round, to attach to the side of the table to give a specific pivot point.
10. A timer device, for example a stopwatch or phone.

The above list indicates some common possibilities. You are encouraged to use your imagination and to improvise as needed alternative ways of building an effective apparatus using the materials that are available to you. Figure 1 shows an example home-made apparatus.

3.3. Prelab Submission

Please respond to the following questions and submit your written answers by the deadline given in class.

1. The derivation given in the Theory section uses mass-based coordinates to derive the matrix equation of motion Eq.6. This gives a diagonal mass matrix. Instead, use string-based coordinates x_1 and x_2 to formulate an equivalent equation of motion. This will produce a diagonal stiffness matrix. Make a second copy of the Matlab script shown in the Appendix and replace the two matrix definition lines **`M = ...`** and **`K = ...`** with your string-based results. Conversion from mass-based to string-based coordinates is no longer needed, so the subsequent line **`Vx = [[1 1]'  [-a1  a2]'] * V;`** should be replaced with **`Vx = V;`** Demonstrate that your string-based analysis gives the same results as the mass-based analysis.

2. The center of percussion concept has a reciprocal character, that is, if an object rotating around Point A has a center of percussion at Point B, then the same object rotating around Point B would have a center of percussion at Point A. Demonstrate this relationship theoretically.
3. Gather the materials listed in Section 3.2 and submit a photo of them. Focus on the main items; do the best you can but not get upset if you should subsequently discover that you also need some additional small items.

4. Introduction

Practical vibrating systems typically have multiple degrees-of-freedom (DOF). For each degree-of-freedom there is a natural frequency and a vibration mode shape. The bifilar pendulum considered here is a multi-DOF vibrating system that has important practical application for experimental determination of the radius of gyration of irregular shaped objects, subsequently giving moment of inertia when mass information is included. On the theoretical side, a bifilar pendulum can be analyzed algebraically and has several interesting and revealing behaviors.

A bifilar pendulum differs from a conventional pendulum in that it has two strings (Latin: "bi" = two, "filum" = a string). Figure 1 shows a typical arrangement. Most commonly, the two strings are adjusted to have equal lengths and to be parallel to each other, but these are not essential requirements. The bifilar pendulum has 3 DOF, which are transverse motions in the x and y directions shown in Fig.1 and a rotational motion around the z axis. For the case of two equal-length parallel strings, these three motions act independently as separate vibration modes. The translation vibration modes resemble two simple pendulums joined together, moving in either the x or y directions. Since the natural frequency of a simple pendulum depends only on string length (and "g"), independent of pendulum mass, the two equal strings of a bifilar pendulum effectively combine to form a simple pendulum spread out over a bigger area. Thus, for a given string length, the translational natural frequencies of a bifilar pendulum and a simple pendulum are the same.

The third vibration mode, corresponding to rotation around the z axis, is the one of greatest practical interest. It will be shown in the Theory section that the rotational natural frequency depends directly on the radius of gyration of the suspended object. (Recall: moment of inertia = mass × radius of gyration^2). The rotational natural frequency of a bifilar pendulum depends specifically on the radius of gyration of the suspended object, not the moment of inertia. Similar to a simple pendulum, the natural frequency is independent of mass.

In this experiment you will measure the natural frequencies and mode shapes of a bifilar pendulum and investigate how they vary with changes in geometrical dimensions. On the way you will learn about the Center of Percussion of an object. You will also use the bifilar arrangement to measure the radius of gyration of some example objects. Finally, you will explore the very unusual natural frequencies and mode shapes of a bifilar pendulum with unequal string lengths.

5. Theory

5.1. Vibration Equations

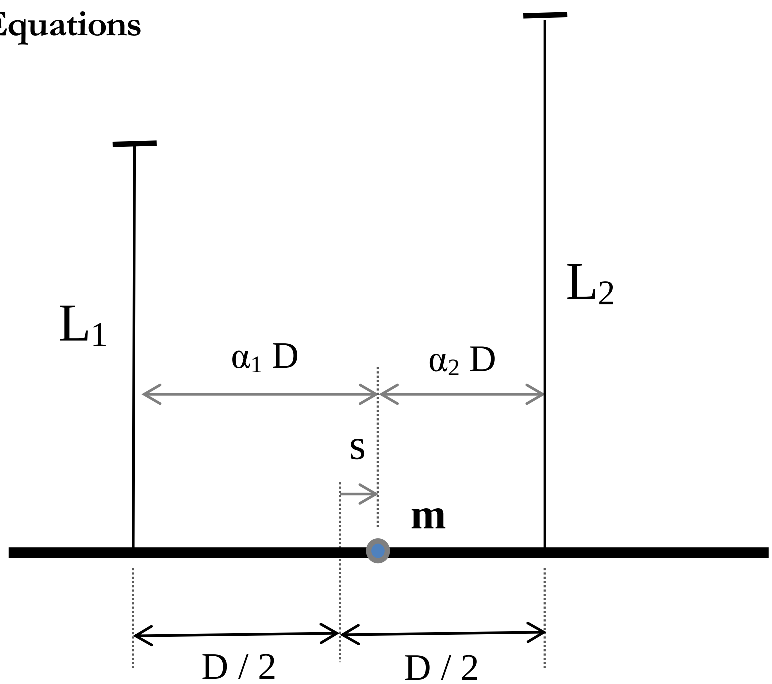

Figure 2. Side view of a bifilar pendulum suspended on strings of lengths L_1 and L_2. The solid dot indicates the center of mass of the suspended rod.

Figure 2 shows a side view of a bifilar pendulum suspended on strings of lengths L_1 and L_2. The center of mass of the suspended rod, indicated by the solid dot, is offset from the centerline of the two strings by a distance s. The center of mass therefore divides the string separation distance D by fractions:

$$\alpha_1 = ½ + s/D \quad \text{and} \quad \alpha_2 = ½ - s/D, \quad \text{where} \quad \alpha_1 + \alpha_2 = 1 \tag{1}$$

The weight of the suspended rod, mg, is supported by the tension forces in the two strings, which are divided in the opposite fractions:

$$F_1 = \alpha_2\, mg \quad \text{and} \quad F_2 = \alpha_1\, mg \tag{2}$$

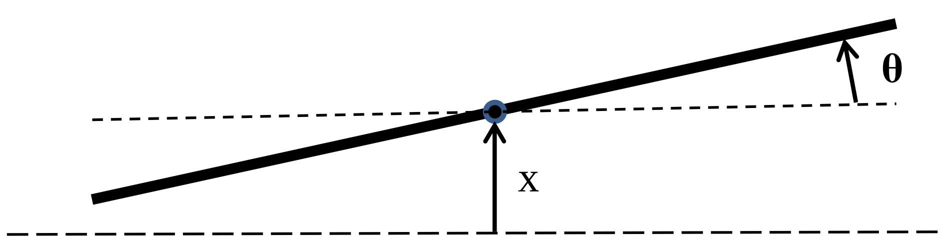

Figure 3. Plan view of the bifilar pendulum showing the displacement coordinate system based on the center or mass.

To create a diagonal mass matrix in the subsequent formulations, a vibrational displacement coordinate system based on the center of mass is chosen. (Usually, a diagonal mass matrix is algebraically easier to handle, here it will turn out to be ideal). Figure 3 shows the chosen coordinate system within a plan view of the bifilar pendulum.

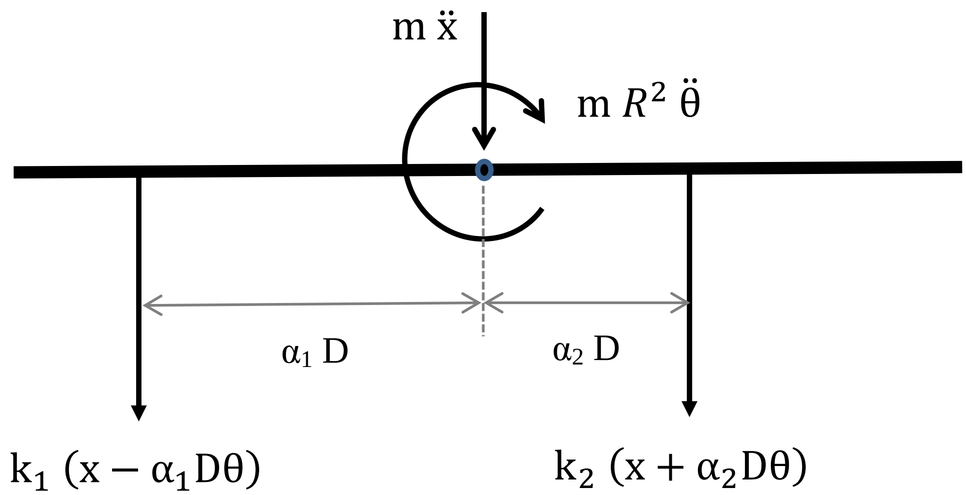

Figure 4. Plan-view free-body diagram of the bifilar pendulum

Figure 4 shows a plan-view free-body diagram of the bifilar pendulum. The D'Alembert method has been used to represent the inertia forces and moments, thereby reducing the dynamics problem to an equivalent statics problem. The rotational moment of inertia I of the pendulum rod is represented as mR^2, where R is the radius of gyration. To simplify the algebra, the horizontal reaction forces at the strings are represented in terms of equivalent spring stiffnesses k, each depending the string tension force and length. Thus:

$$k_1 = \frac{F_1}{L_1} = \frac{\alpha_2 \, mg}{L_1} \qquad\qquad k_2 = \frac{F_2}{L_2} = \frac{\alpha_1 \, mg}{L_2} \tag{3}$$

A force balance in the downward vertical direction gives:

$$m\ddot{x} + k_1(x - \alpha_1 D\theta) + k_2(x + \alpha_2 D\theta) = 0 \tag{4}$$

and a moment balance in the clockwise direction gives:

$$mR^2 \ddot{\theta} - k_1(x - \alpha_1 D\theta)\alpha_1 D + k_2(x + \alpha_2 D\theta)\alpha_2 D = 0 \tag{5}$$

Substituting for k_1 and k_2, rearranging and putting into matrix form gives:

$$\begin{bmatrix} m & 0 \\ 0 & m\frac{R^2}{D^2} \end{bmatrix} \begin{bmatrix} \ddot{x} \\ D\ddot{\theta} \end{bmatrix} + \frac{mg\,\alpha_1\alpha_2}{L_1L_2} \begin{bmatrix} \frac{L_1}{\alpha_2} + \frac{L_2}{\alpha_1} & L_1 - L_2 \\ L_1 - L_2 & \alpha_2 L_1 + \alpha_1 L_2 \end{bmatrix} \begin{bmatrix} x \\ D\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6)$$

where the rotational coordinate has been multiplied by D to make the matrices dimensionally consistent.

For equal length strings, $L_1 = L_2 = L$. Substituting $\alpha_1 + \alpha_2 = 1$ gives:

$$\begin{bmatrix} m & 0 \\ 0 & m\frac{R^2}{D^2} \end{bmatrix} \begin{bmatrix} \ddot{x} \\ D\ddot{\theta} \end{bmatrix} + \frac{mg}{L} \begin{bmatrix} 1 & 0 \\ 0 & \alpha_1\alpha_2 \end{bmatrix} \begin{bmatrix} x \\ D\theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (7)$$

The matrices in Eq.7 are both diagonal, indicating that x and θ are Principal Coordinates. The equations are uncoupled and may be solved individually.

Vibration mode 1 is a pure translation in the x direction. The bifilar pendulum swings from side to side without rotation as if it were a simple pendulum. The natural frequency $\omega_1^2 = g/L$ corresponds to that of a simple pendulum.

Vibration mode 2 is a pure rotation around the center of mass, for all values of the string offset s. After substituting Eq.1, the natural frequency in is:

$$\omega = \frac{D}{R}\sqrt{\frac{g}{L}}\sqrt{\frac{1}{4} - \left(\frac{s}{D}\right)^2} \quad (8)$$

For the symmetrical case where s = 0:

$$\omega = \frac{D}{2R}\sqrt{\frac{g}{L}} \quad (9)$$

5.2. Special Case: Center of Percussion

In general, the two natural frequencies of the bifilar pendulum are different, each with its own specific mode shape. Vibration can be in one mode or other, or in both simultaneously. When combined vibration occurs, the motion deviates from simple harmonic, and appears irregular. An interesting special case occurs when $\omega_1 = \omega_2$, for which:

$$D^2 = \frac{R^2}{\alpha_1\alpha_2} \quad (10)$$

In this case, the two same-frequency vibrations come together smoothly to form a combined mode shape that retains simple harmonic motion. If the translational and rotational mode shapes shown in Fig.5(a) and (b) are added, they form the combined mode shape shown in Fig.5(c). The combined mode shape has a nodal point at the first string position. If the second mode shape were subtracted from the first, a mode shape with a nodal point at the second string position would be obtained. All these vibration modes have natural frequency $\omega = \sqrt{(g/L)}$.

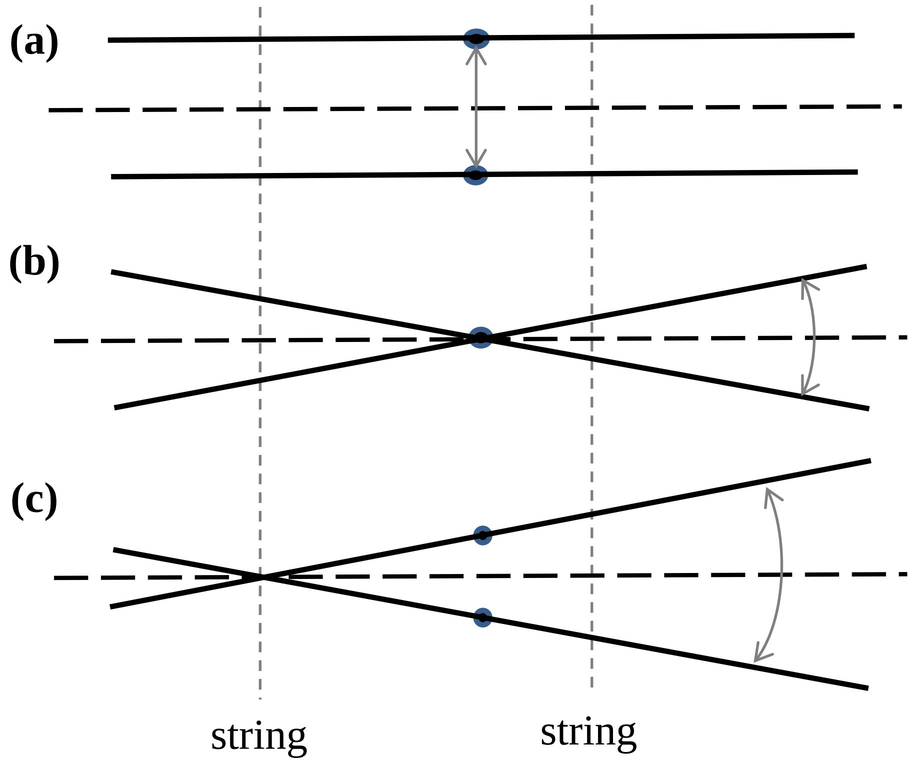

Figure 5. Plan view of mode shapes of a bifilar pendulum for which $\omega_1 = \omega_2$. Any combination of mode shapes is a valid mode shape. The vertical lines indicate the string positions and the dots the center of mass. (a) translational (simple pendulum) vibration, (b) rotational vibration, (c) combination vibration with a nodal point at a string location.

The mode shape in Fig.5(c) is remarkable because the presence of a nodal point at one of the string positions means that there is no string motion there and hence no resultant force. All the reaction force from the vibration is taken by the second string. The position of the second string is said to be at the "Center of Percussion" for rotations around the first string position. The center of percussion concept is familiar to baseball and tennis players. There is a "sweet spot" on the bat or racquet at which the ball hits smoothly without producing any reaction force on the player's hand. This "sweet spot" corresponds to the center of percussion.

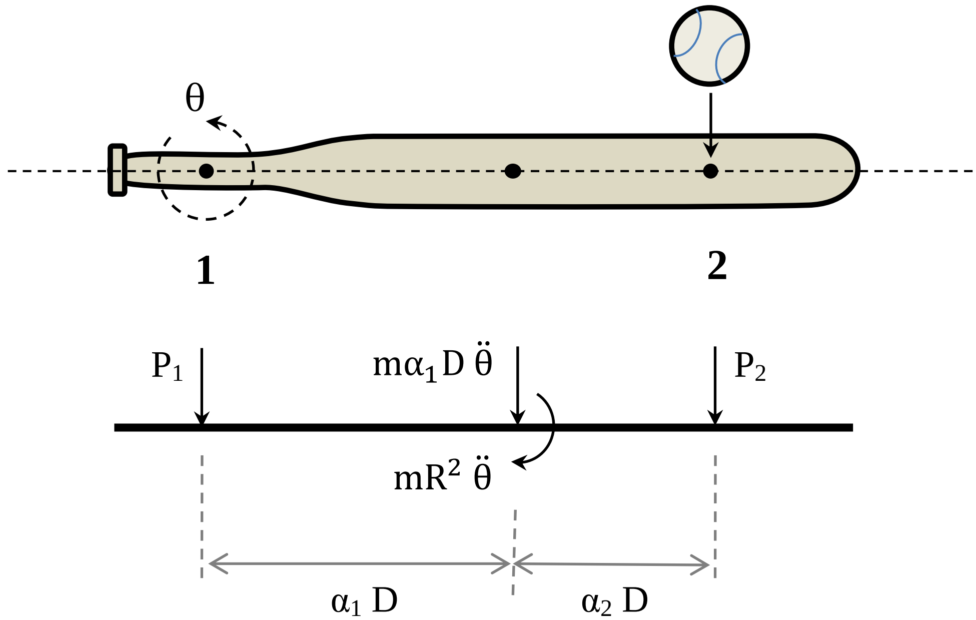

Figure 6. Impact forces on a baseball bat.

The free-body diagram of the baseball bat in Fig.6 represents the case where a batsman rotates the bat around Point 1 and hits the ball at Point 2. If Point 2 is the "sweet spot", then the reaction force P_1 at Point 1 is zero. Taking moments about Point 2 gives:

$$m\,\alpha_1 D\ddot{\theta}\alpha_2 D - mR^2\ddot{\theta} + P_1 D = 0 \tag{11}$$

For $P_1 = 0$:

$$D^2 = \frac{R^2}{\alpha_1\alpha_2} \tag{12}$$

which is the same as Eq.10. Thus, we see that the equal natural frequency condition $\omega_1 = \omega_2$ occurs when one string of the bifilar pendulum at the center of percussion of the other.

The center of percussion is an important concept beyond its use in sports. For example, the wheels of many cars are designed so that the front and back wheels are at the centers of percussion relative to each other. In that way, the forces from road bumps felt by the front wheels are absorbed smoothly and do not transfer to the back wheels, and vice versa.

5.3. Unequal String Lengths

The most common practical use of a bifilar pendulum is to measure the radius of gyration of a test object. In that case, it is usual to arrange both strings to have the same length. However, from the point of view of vibration behavior, it is very interesting to explore the response of a bifilar pendulum with unequal string lengths. In that case, the full matrix Eq.6 applies. The equation is coupled and is not convenient to solve algebraically, thus a numerical approach is taken here. The Matlab script shown in the Appendix uses routine "eig" to solve the generalized eigenproblem

$$\mathbf{K\,u} = \omega^2\,\mathbf{M\,u} \tag{13}$$

where **M** and **K** respectively are the mass and stiffness matrices shown in Eq.6 and **u** is the mode shape vector. **Bold** font indicates matrix and vector quantities. The eigenvalues are the squared natural frequencies and the eigenvectors are the mode shapes.

Useful insight can be obtained by first considering the case where the bifilar pendulum strings are at the centers of percussion, which for a symmetrical pendulum corresponds to $D = 2R$. For a uniform rod of length ℓ, the radius of gyration $R = \ell/\sqrt{12}$. As discussed in Section 5.2, a vibrational force at a center of percussion produces no reaction force at the corresponding center of rotation. Thus, the bifilar pendulum can vibrate as if it comprised two independent simple pendulums, each involving movement of one string without movement of the other. The resulting vibrations are said to be "uncoupled." The individual simple pendulum natural frequencies are:

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{g}{L_1}} \quad \text{and} \quad f_2 = \frac{1}{2\pi}\sqrt{\frac{g}{L_2}} \tag{14}$$

Fig.7 shows a graph of f_1 and f_2 vs. length ratio L_2/L_1. The first natural frequency f_1 is independent of L_2/L_1, and so remains constant. The second natural frequency decreases with L_2/L_1, according to an inverse square-root relationship. The two lines cross at $L_1 = L_2$.

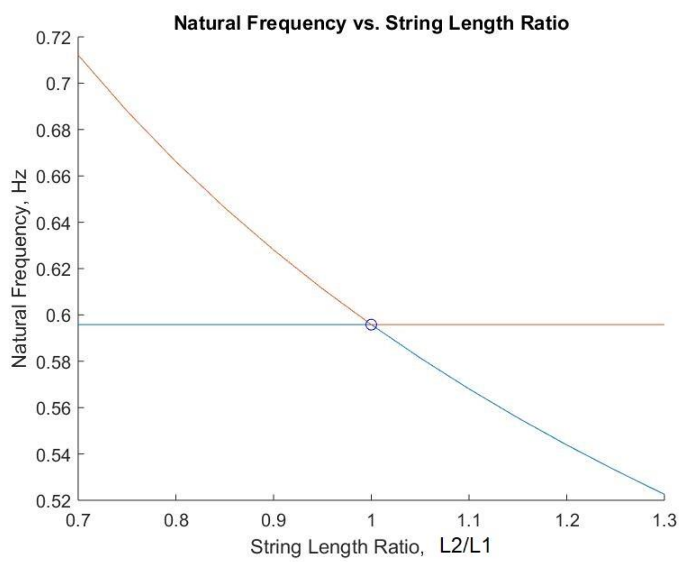

Figure 7. Natural frequency vs. string length ratio for a bifilar pendulum with symmetrical string separation $D = 2R$. String length $L_1 = 0.70$m.

A change in string separation away from $D = 2R$ disrupts the exact center of percussion relationship of Fig.7 and causes each string motion to affect the other. The string motions are then said to be "coupled". Figure 8(a) shows the natural frequency vs. string length ratio for a bifilar pendulum with $D = 2.05$ R. A remarkable thing happens;

instead of the two lines crossing over each other, as in Fig.7, they start to approach, but then veer away to follow the path of the other line. This is called an "avoided crossing", also sometime described as "curve veering". Before the avoided crossing, the vibration mode mostly consists of the corresponding vibration mode from Fig.7. However, while traversing the avoided crossing there is a rapid transformation to the second vibration mode on the other side. Within the avoided crossing, the vibration comprises a mixture of the two vibration modes. The small circle at $L_2/L_1 = 1$ corresponds to case where the entire bifilar pendulum moves side to side as a simple pendulum with natural frequency $f = \sqrt{(g/L_1)}/2\pi$. One of the two lines always passes through this point; for $D > 2R$ it is the lower line, for $D < 2R$ it is the upper line.

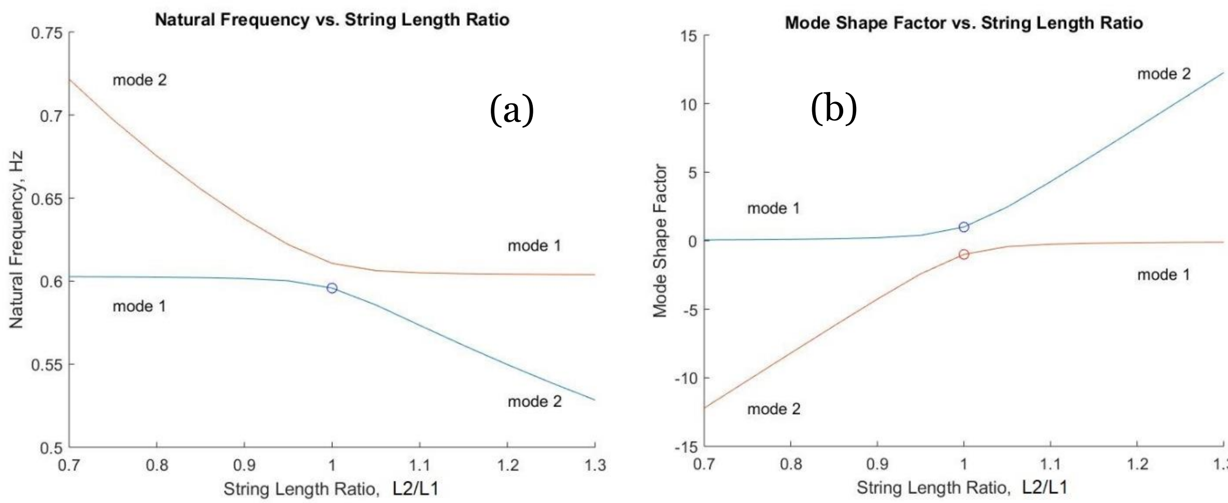

Figure 8. Vibration response vs. string length ratio for a bifilar pendulum with symmetrical string separation D=2.05R. String length $L_1 = 0.70$m. (a) natural frequencies, (b) mode shape factors.

Figure 8(b) shows the corresponding change in the mode shape. Since the interest here is on the motions of the two strings, the displacement coordinate system has been changed from the center-of-mass based arrangement shown in Fig.3 to a coordinate system $[x_1 \; x_2]^T$ based on the lateral displacements at the two string locations. The transformation between the two coordinate systems is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -\alpha_1 \\ 1 & \alpha_2 \end{bmatrix} \begin{bmatrix} x \\ D\theta \end{bmatrix} \tag{15}$$

The "mode shape factor" is the ratio x_2/x_1. Along the horizontal lines in Fig.8(b) the ratio x_2/x_1 is much less than 1, indicating that x_1 is much bigger than x_2. Thus, these lines indicate a vibration mode that mainly involves String 1, i.e., "mode 1". Conversely, along the diagonal lines in Fig.8(b), the ratio x_2/x_1 is much larger than 1, indicating that x_2 is much bigger than x_1. Thus these lines indicate a vibration mode that mainly involves

String 2, i.e., "mode 2". In the middle area, within the avoided crossing, the ratios x_2/x_1 have values around 1, thus indicating vibration modes involving both strings. The small circles at $L_2/L_1 = 1$ correspond to the symmetrical case where the vibration modes are pure translation and pure rotation about the center or mass, as shown in Fig.5(a) and (b), with x_2/x_1 values of ±1. All mode shape factor curves pass through these two points, independent of string spacing D.

Figure 9 shows the similar curves to Fig.8 for the case D = 2.10R, which is a yet further away from the Center of Percussion case D = 2R. It can be seen that the avoided crossings still exist, but are smoother and have progressed further away from their original crossing points. Adjustment of D yet further away from 2R will cause the lines to become yet smoother and more separated.

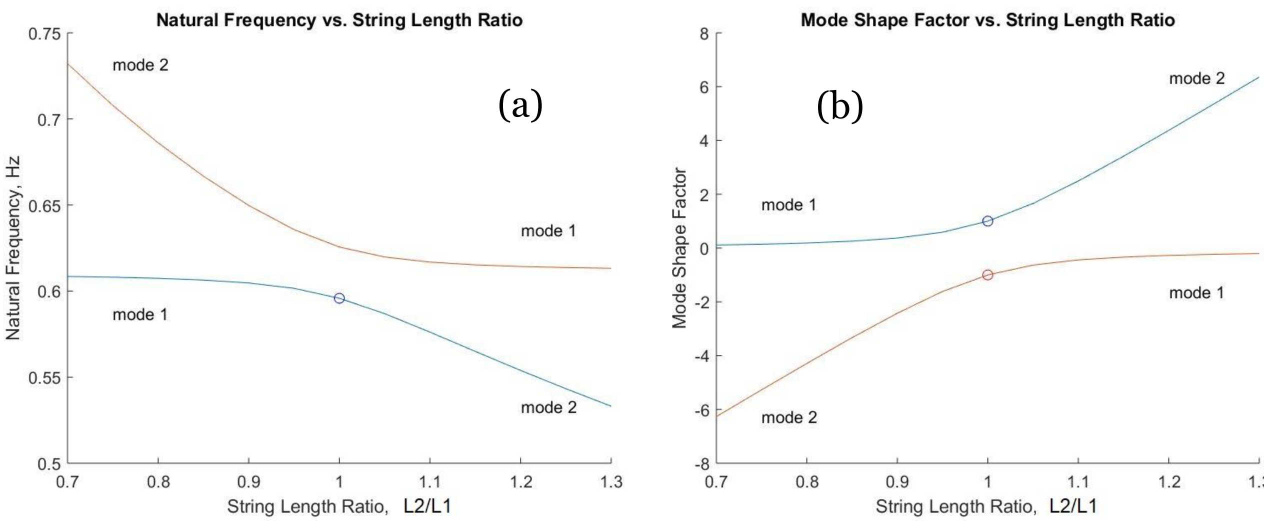

Figure 9. Vibration response vs. string length ratio for a bifilar pendulum with symmetrical string separation D=2.10R. String length L_1 = 0.70m. (a) natural frequencies, (b) mode shape factors.

In addition, it can be seen that the vertical scale of the mode shape factor graph in Fig.9(b) has reduced, indicating that the string displacements become less separate as D differs more from 2R. For the D = 2R case considered in Fig.7, the string displacements are entirely separate. If the mode shape factor were instead computed for 2.0000001R, then the Matlab script in the Appendix would produce a mode shape factor plot similar to Fig.7, but with a massive vertical scale numbered in the millions. No mode shape factor plot is possible for the case D = 2R exactly because "infinite" results are produced.

The avoided crossing phenomenon occurs quite often in vibrating systems. Figure 10 shows an example of the natural frequencies of a rotating circular string, used as a model for a rotating circular saw. It can be seen that the mode shapes, as indicated by the adjacent numbers, vary as each line passes through an avoided crossing.

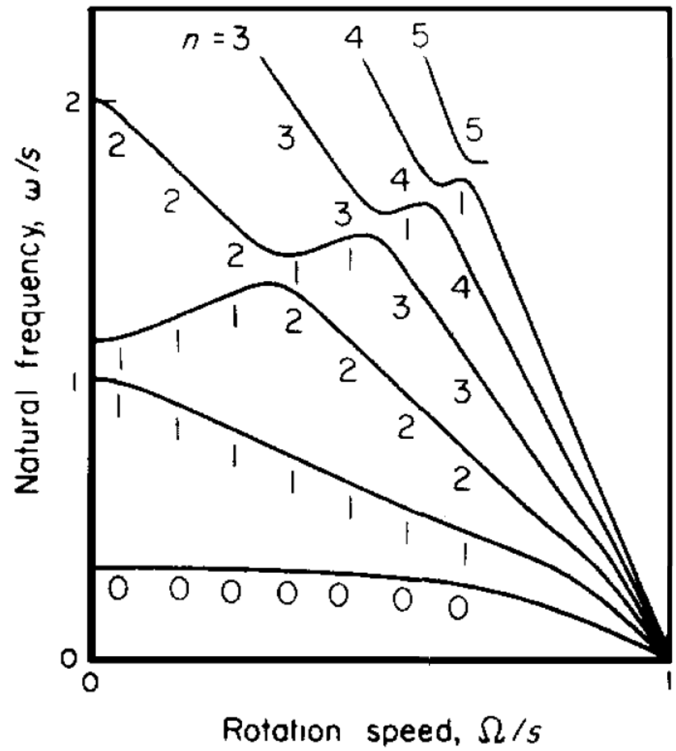

Figure 10. Natural frequencies vs rotation speed of a guided rotating circular string. The digits indicate the numbers of nodal diameters in the vibration mode shapes. (Adapted from "The Vibration of a Rotating Circular String Subject to a Fixed Elastic Restraint", Schajer, G.S., J. Sound and Vibration, Vol.92, No.1, pp. 11-19, 1983.)

6. Experiment Procedure

Assemble your bifilar pendulum to resemble the arrangement shown in Fig.1. Insert a pencil at the upper end position of the String 1 (L_1) to create a distinct pivot point. Tie loops at each end of a length of dental floss, the lower loop to support the pendulum stick and the upper loop arranged with some adhesive tape running through it. That tape can then be stuck to the table at an appropriate place to make the ruler/stick hang a few cm above the floor. This will maximize your pendulum length and give it an easily measured period of oscillation.

The connection to String 2 (L_2) is more complex because it is desired to be able to change this string length later in the experiment. Prepare your packing box by filling it with heavy books so that it has significant weight, then seal the box to stiffen its structure. Fold each of the two cardboard pieces 50cm x 6cm in half lengthways to form 90deg angle beams. Tape the two pieces together front and back to form a sturdy T-beam and then tape the T-beam onto the side of the packing box, as shown in Fig.12. Be careful to tape the T-beam to the box securely so as to create as rigid a connection as possible. It will be useful to slope the T-beam slightly outwards towards the bottom so as to provide some extra space for the string to swing freely. If your box wobbles slightly, slide some extra cardboard strips under the edges as needed to make the box firm. Use a binder clip to secure the upper end of String 2. You can then adjust the pivot point of the string by moving the clip up or down along the T-beam. Concurrently, you can adjust the length of the string by winding excess string around the binder clip until the pendulum stick is suspended horizontally. Figure 11 shows an example setup, feel free to adapt the design according to the materials that you have available.

Orient your stick so that the narrow side is vertical; this reduces wind resistance and minimizes damping. If the length of your stick is different from 1m, scale all dimensions mentioned in the following descriptions by the ratio (stick length in m / 1m). Make the measurements described in the next subsections, record your measurements neatly and systematically in your lab book, and be sure to take some photos to include in your report

Figure 11. Example setup for an adjustable-length support for String 2.

6.1. Simple Pendulum Vibration

1. Adjust position of the binder clip such that the lengths of the two strings are equal. Set the string separation "D" = 90cm, with the stick center of mass midway between. Measure the string separation at both top and bottom ends of the strings to ensure that they are accurately parallel. Adjust the box position as needed. Also carefully measure and record the lengths of your strings. Ensure that they are equal. (Hint: measure from the pivot point down to the center thickness of your stick.). The different mounting arrangements for the two strings will cause the stick to hang not exactly parallel to the table edge. This is not a problem because all that is necessary is for the strings to be parallel to each other.
2. The two lateral vibration modes of the bifilar pendulum in the x and y directions resemble that of a simple pendulum, so their natural frequencies should be the same, independent of string separation. Displace the stick in the y (longitudinal) direction and let go. Check the subsequent vibrations to ensure that they are purely longitudinal and that the strings are not clashing with the support structure. When satisfied, use your timer to measure the time interval for 10 full oscillations. (Hint: put a small object to use as a marker near the center of oscillation and measure the oscillation time starting and finishing when the stick passes past the marker. This works better than timing at the ends because the stick moves the fastest at the center, so the time boundaries can be seen more distinctly.) Take the reciprocal of the vibration period to determine the natural frequency.

3. Using a metal object such as the back of a spoon, tap the stick at its center in the x (lateral) direction. This will start pure lateral oscillations. Use your timer to time 10 oscillations. The measured natural frequency should be the same as before. Compare the result with the expectation from a simple pendulum.

6.2. Effect of String Separation

4. Gently touch your finger underneath the center of your stick so as to hold it in place while displacing one end of the stick with a finger of the other hand. This will cause the stick to rotate around your first finger position. Then simultaneously let go with both fingers to start the stick vibrating around the center position. Allow the stick to oscillate a few times so you can confirm that it is accurately rotating around the center position, without any side-to-side motion. You may need to practice a few times to perfect the procedure. When ready, time 10 oscillations and find the natural frequency.
5. Repeat Step 4 with the strings adjusted symmetrically to D = 80cm, 70cm, 60cm 50cm, 40 cm. As before, measure at the top and bottom of the strings to endure that they are parallel. At each setting, measure 10 rotational oscillations and determine the natural frequency. There is no need to repeat Step 3 at each setting because the frequency of lateral vibrations should be independent of string separation.
6. Plot a graph of measured frequency f (Hz) vs string separation D. This should be a straight line. Evaluate its gradient G (remember to work in terms of m, not cm). Determine the radius of gyration R of the stick using the formula

$$R = \sqrt{\frac{g}{L}}\frac{1}{4\pi G} \tag{16}$$

6.3. Effect of Stick Offset

7. Reset the string separation D to 50cm. Set the center of the stick midway between the strings and measure 10 rotational oscillations. This corresponds to s = 0.
8. Displace the stick lengthways between the strings by s = 5cm. Measure 10 rotational oscillations. (Hint: the nodal point remains at the center of mass, so you can continue to use the finger-touch technique at the stick center.)
9. Repeat Step 8 for stick offsets s = 10cm, 15cm, 20cm and 22cm.
10. Plot a graph of measured frequency f (Hz) vs normalized string separation s/D.
11. Compute the theoretical natural frequency F using

$$F = \frac{D}{2\pi R}\sqrt{\frac{g}{L}}\sqrt{\frac{1}{4} - \left(\frac{s}{D}\right)^2} \qquad (17)$$

And plot the theoretical frequency F on the same graph as the measured frequencies f. How well do the results compare ?

6.4. Center of Percussion Suspension

12. Symmetrically place your stick between the strings and set the string separation to D = 57.4cm (or scaled distance if your stick is not exactly one metre long). Use the finger touch technique to start the stick vibrating around the center of mass and measure 10 rotational oscillations. Compare the associated vibration frequency with that observed in Step 3; it should be the same.
13. Move your attention to String 1 and use the finger touch technique to start the stick vibrating around the string connection. If all goes well, String 1 will remain still while only String 2 moves. String 2 is then at the center of percussion for rotations around String 2. Time 10 oscillations; the associated frequency should again be the same.
14. Confirm that String 2 is the center of percussion relative to String 1 by tapping the stick at the String 2 position and observing that the resulting vibration has a nodal point at String 1.
15. Repeat Steps 13-14 for rotations around String 2.
16. Repeat Steps 13-14 for rotations and hits at any other arbitrary point along the length of the stick. For all Steps 12-15, you should see sinusoidal vibrations that are smooth but whose nodal positions gradually drift around. Why does this happen?

6.5. Unequal String Lengths

17. Symmetrically place your stick between the strings and set the string separation to D = 60cm (or scaled distance if your stick is not exactly one metre long). Adjust the box position so that the strings are parallel. Set the string length $L_2 \approx 0.7L_1$. and record the lengths of both strings.
18. Set the bifilar pendulum vibrating in its lower frequency vibration mode and measure the time for 10 rotational oscillations. You can use the finger touch technique to start the oscillations, using the nodal points indicated in Table 1. Interpolate as needed. You may need to practice this a few times until you can get a clean sinusoidal oscillation. Alternatively, you can tap the stick at the nodal point of the higher frequency vibration mode. Try both ways and choose which one works better for you. When the pendulum vibrates in a single mode it will swing smoothly back and forth in a steady pattern and will have a well-defined nodal point.

Try tapping the stick at a different point and notice that the pendulum motion becomes apparently chaotic (actually a mixture of both vibration modes).

L_2/L_1	**Lower Freq. Nodal point, cm**	**Higher Freq. Nodal Point, cm**
0.70	35.8	– 23.2
0.75	37.7	– 22.1
0.80	40.5	– 20.5
0.85	45.7	– 18.2
0.90	56.9	– 14.6
0.95	95.4	– 8.7
1.00	∞	0.0
1.05	– 99.5	8.4
1.10	– 60.4	13.8
1.15	– 48.9	17.0
1.20	– 43.6	19.1
1.25	– 40.5	20.5
1.30	– 38.6	21.6

Table 1. Distances in cm towards String 2 from center of mass to nodal points, for D = 60cm. The center of percussion of one vibration mode is at the nodal point of the other.

19. Repeat Step 18 to measure the higher frequency vibration mode.
20. Repeat Steps 17 -- 19 for string lengths $L_2 \approx 0.8L_1$, $0.9L_1$, $0.95L_1$, L_1, $1.05L_1$, $1.1L_1$, $1.2L_1$, $1.3L_1$. If the nodal point is not on the stick, then use the tapping technique. Compute the associated natural frequencies and draw a graph vs. L_2/L_1. If all goes well, you should get a graph similar to Fig.9.

6.6. Radius of Gyration Measurement

21. Select a regularly shaped object that you have in your home, for example a kitchen cutting board, a hardcover book or a plate. Suspend it on two equal length strings to form a bifilar pendulum, measure the string lengths and separation, and then the measure the time for 10 rotational oscillations. Put G = f/D and compute the radius of gyration using Eq.16. Compare the measured result with the theoretical expectation based on geometry.

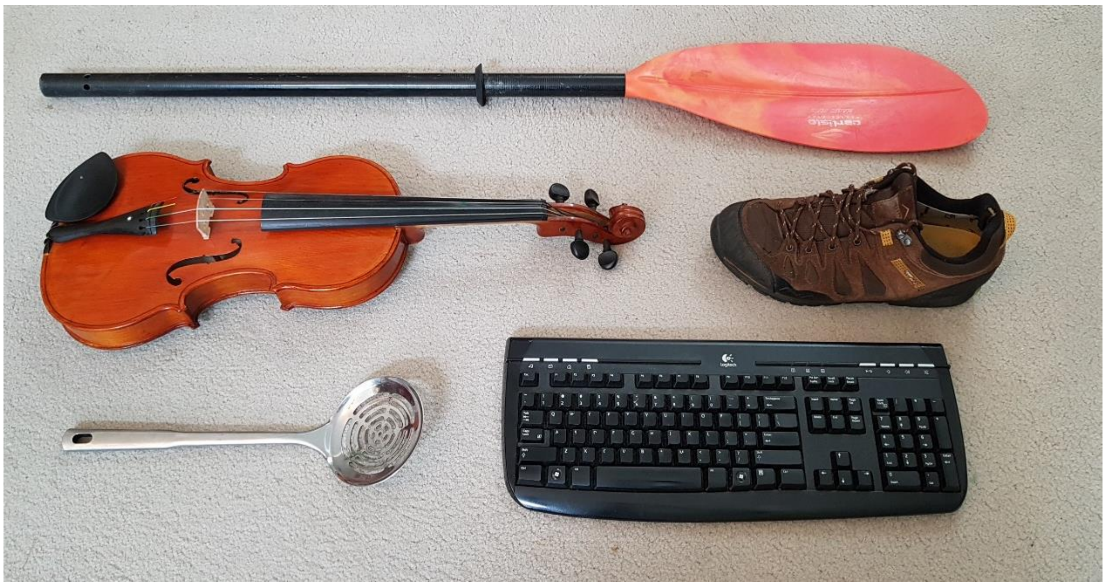

Figure 12. Some example irregular-shaped objects that could be a suitable specimen for a radius of gyration measurement.

22. Select an irregularly shaped object that have in your home, preferably with approximate left-to-right symmetry so that it will hang horizontally when suspended. Look around and use your imagination to find something interesting. Fig.12 shows some example things I found in my own home. Support the object on one finger and mark the balance point (center of mass). Suspend the object between two strings, measure the string length L, string separation D, and offset of the center of mass s. Avoid s/D > 0.2 by moving the "far" string inwards. Then measure the time for 10 rotational oscillations as before. Find the radius of gyration from Eq.16 (rearrange it to bring R to the left). Approximate the shape of your object as a combination of some basic shapes and estimate its radius of gyration. Compare that result with your measurement.

7. Post-Lab Question

The experiment here used parallel strings throughout. What effect on natural frequencies would you expect if the strings were non-parallel (separation between the top supports greater or lesser than the lower connections to the pendulum) ? Describe the expected effects and explain your reasoning. For simplicity, you may assume that the string lengths are the same.

8. Report Requirements

Prepare a concise professional report that includes all needed points and that excludes unessential material. A moderately formal presentation is appropriate, with sections as follows:

- **Introduction.** Introduce the topic of your report. Give a brief background and explain needed concepts. Keep things short; there is no need to reproduce all the theoretical details from this lab instruction manual. Open the issues that you will address during your investigations and on which you will present your conclusions at the end. (Note that the Introduction and Conclusions sections should be "symmetrical", thus the issues raised in the Introduction should be addressed in the Conclusions. Conversely, do not raise any issues in the Introduction that are not addressed in the Conclusions, and vice-versa.)
- **Theory.** Keep this section short, do not grind your way through all the details from the lab instructions. Just mention the key points that are needed to understand what you did during your experiments. You may directly quote the theoretical formulas.
- **Apparatus.** Describe your apparatus and comment on any notable design features, particularly those that provide good functionality. Also, indicate concerns about any adverse features and suggest possible future improvements. Show a modest number of photos to illustrate your descriptions.
- **Procedure and Results.** Describe the key points of your procedure and display your measurements. Focus on the meaning and significance of your results and present a compact "story" of your findings. Be careful, remember that your report is not a diary, it is about your results, not about you.
- **Discussion.** Discuss the important features of your findings, particularly their significance and the connections among them.
- **Conclusions.** Summarize the main conclusions that can be drawn from your investigations. Remember to keep your Introduction and Conclusions "symmetrical".

9. Appendix

The following is an example Matlab code to do the calculations described in the Theory section.

```
% Bifilar Pendulum: Natural Frequencies and Mode Shapes

% Author:  Gary S. Schajer, 2020.

% Variables:
% ----------
%  a1 = offset ratio 1
%  a2 = offset ratio 2
%  cp = center of percussion position, m
%  D  = distance between strings, m
%  f  = natural frequency, Hz
%  g  = gravitational acceleration = 9.81 m/s^2
%  i  = index for length ratio plotting
%  j  = index for eigenvectors
%  K  = stiffness matrix, N/m
%  L1 = string length 1, m
%  L2 = string length 2, m
%  LL = pendulum rod length, m
%  LR = length ratio = L2/L1
%  m  = pendulum mass, kg
%  M  = mass matrix, kg
%  np = nodal point position, m
%  R  = radius of gyration of pendulum rod, m
%  s  = pendulum centroid offset, m
%  v  = eigenvector ratio, V(2,:)/V(1,:)
%  V  = eigenvector matrix
%  vx = string displacement eigenvector ratio
%  Vx = string displacement eigenvector matrix
%  w2 = angular frequency squared, (rad/s)^2

%**********************************************

% Initialize variables
clear all;
close all;
m = 1;   % any positive number will work
g = 9.81;
LL = 1;
R = LL/sqrt(12);
L1 = 0.70;
L2 = 0.70;
D = 0.60;   % = 2.0000001*R;  for Fig. 7
s = 0;
a1 = 0.5 + s/D;
a2 = 0.5 - s/D;

% Solve for length ratios in the range 0.7 to 1.3
for i = 1:13
  L2 = L1 * (0.65 + 0.05*i);
  LR(i) = L2 / L1;
  M = [[m 0]' [0 m*R^2/D^2]'];
  K = m*g*a1*a2/L1/L2 * [[L1/a2+L2/a1 L1-L2]' ...
                         [L1-L2 a2*L1+a1*L2]'];
```

```
  [V,w2] = eig(K,M,'vector');
  [w2,index] = sort(w2);
  V = V(:,index);
  f(i,:) = sqrt(w2) / 2 / pi;

  % Find mode shape ratio and nodal point position
  v(i,:) = V(2,:) ./ V(1,:);
  Vx = [[1 1]' [-a1 a2]'] * V;
  Vx(:,:) = Vx(:,:) ./ norm(Vx(:,:));
  vx(i,:) = Vx(2,:) ./ Vx(1,:);
  np(i,:) = (a2*Vx(1,:)+a1*Vx(2,:))*D./(Vx(1,:)-Vx(2,:));
  if abs(np(i,1)) > 10
    np(i,1) = NaN;
  end
  cp(i,:) = -R^2 ./ np(i,:);
end

% Draw nodal position vs length ratio plot
figure(3)
hold on
plot(LR,np)
plot(1, 0, 'or')
text(LR(2),np(5,1),'mode 1')
text(LR(2),np(6,2),'mode 2')
text(LR(11),np(9,1),'mode 2')
text(LR(11),np(8,2),'mode 1')
title('Nodal Position vs. String Length Ratio')
xlabel('String Length Ratio,  L2/L1')
ylabel('Nodal Position,  m')

% Draw mode shape factor vs length ratio plot
figure(2)
hold on
plot(LR,vx)
plot(1,  1, 'ob')
plot(1, -1, 'or')
text(LR(2),vx(8,1),'mode 1')
text(LR(2),vx(1,2),'mode 2')
text(LR(11),vx(6,2),'mode 1')
text(LR(11),vx(13,1),'mode 2')
title('Mode Shape Factor vs. String Length Ratio')
xlabel('String Length Ratio,  L2/L1')
ylabel('Mode Shape Factor')

% Draw frequency vs length ratio plot
figure(1)
hold on
plot(LR,f)
plot(1, sqrt(g/L1)/2/pi, 'ob')
text(LR(2),f(8,1),'mode 1')
text(LR(2),f(1,2),'mode 2')
text(LR(11),f(6,2),'mode 1')
text(LR(11),f(13,1),'mode 2')
title('Natural Frequency vs. String Length Ratio')
xlabel('String Length Ratio,  L2/L1')
ylabel('Natural Frequency, Hz')
```